The DTFT analysis and synthesis equations are

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi}\int_{<2\pi>} X(e^{j\omega})e^{j\omega n}d\omega$$

where $< 2\pi >$ is any contiguous interval of length 2π, chosen based on whatever is most convenient for the problem at hand.

The CTFT analysis and synthesis equations are

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

This worksheet will help you explore the Fourier Transform and its properties from the two main perspectives it is used for:

1. Calculating transform pairs to find the frequency content of signals.
2. As a tool for LTI system analysis.

The mechanics are more or less the same in DT and CT. For example, the process of using Fourier Transform properties to solve for more complicated transform pairs is essentially the same for the CTFT as the DTFT, the properties are just slightly different. So, to avoid redundancy, we'll see some ways use case (1) comes up in DT through the first half of the worksheet. Then, we'll work through some examples of use case (2) in CT through the second half of the worksheet.

Problem 1: DTFT Pairs via Brute Force Compute the Fourier Transform (if given $x[n]$) or inverse Fourier Transform (if given $X(e^{j\omega})$). You should use the analysis and synthesis equations for this question.

(a) $x[n] = \delta[n - n_0]$.

(b) $x[n] = \alpha^n u[n]$, where $|\alpha| < 1$.

(c) $X(e^{j\omega}) = \cos(a\omega)$, where a is an integer.

(d) $X(e^{j\omega}) = \sin(b\omega)$, where b is an integer.

(e) Let $\omega_0 < \pi$. For $|\omega| \leq \pi$, the spectrum is given as $X(e^{j\omega}) = \begin{cases} 1 & \omega \in [-\omega_0, \omega_0] \\ 0 & \text{otherwise} \end{cases}$ and 2π-periodically repeats outside this region.

Problem 2: DTFT Properties for Transform Pairs For each of the following signals or spectra, (i) identify the property or properties that allow you to reduce this to a transform pair you have already calculated on this worksheet, and, (ii) compute the DTFT if given $x[n]$ or inverse DTFT if given $X(e^{j\omega})$.

(a) $x[n] = \sum_{k=-N}^{N} \delta[n-k]$. Your answer should be of the form

$$X(e^{j\omega}) = \frac{\sin(a\omega)}{\sin(b\omega)}$$

where a, b are constants you will determine. You may find the identity

$$\sum_{k=C}^{D} a^k = \frac{a^C - a^{D+1}}{1-a}$$

to be of use.

(b) $x[n] = \beta^n u[-n-1]$, where $|\beta| > 1$.

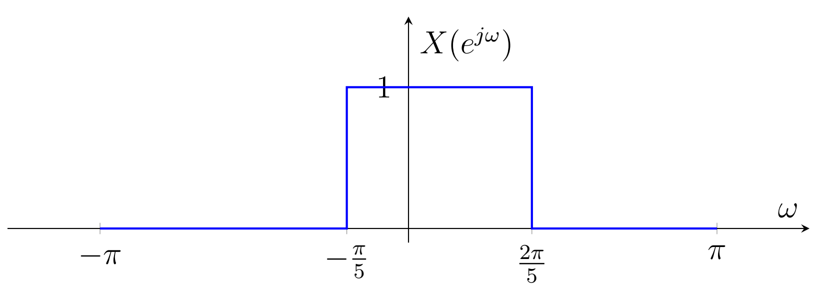
Hint: First, apply a series of transformations to get your signal into the form $\alpha^n u[n]$, where α is some straightforward function of β. Then, backtrack by accounting for the various transformations using Fourier Transform properties to obtain $X(e^{j\omega})$.

(c) $x[n] = \left(\frac{1}{2}\right)^{|n|}$. No need to simplify your answer.

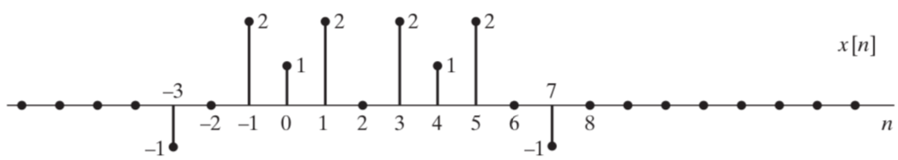
(d) $x[n] = n\left(\frac{1}{2}\right)^{|n|}$.

(e) $X(e^{j\omega}) = \cos(a\omega)\sin(b\omega)$, where a, b are both integers. Note that this assumption on a, b will guarantee a 2π-periodic spectrum. **You do not need any trigonometric identities.**

(f) The spectrum $X(e^{j\omega})$ is shown below for $\omega \in [-\pi, \pi]$, and 2π-periodically repeats outside this region. **You do not need to compute any integrals.**



Problem 3: DTFT Properties for Salient Frequency-Domain Features [1]Let $X(e^{j\omega})$ denote the Fourier Transform of the signal $x[n]$ shown below.



Without computing $X(e^{j\omega})$, determine the following:

(a) $X(e^{j\omega})|_{\omega=0}$.

(b) $X(e^{j\omega})|_{\omega=\pi}$.

(c) $\int_{-\pi}^{\pi} X(e^{j\omega}) d\omega$.

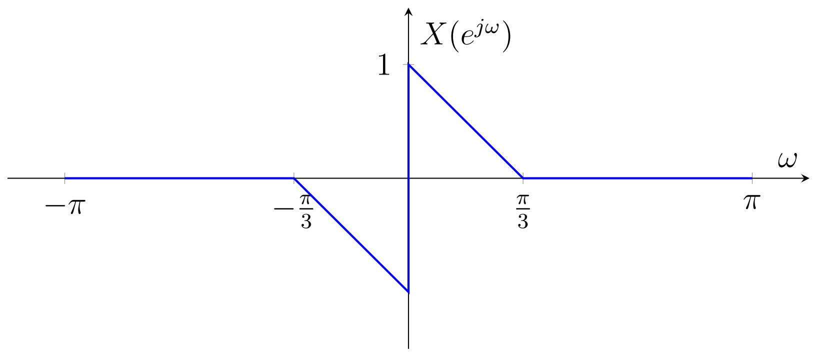
(d) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$.

(e) Is the spectrum conjugate-symmetric, that is, does $X(e^{-j\omega})$ equal $X^*(e^{j\omega})$?

(f) Is the spectrum even-symmetric, that is, does $X(e^{-j\omega})$ equal $X(e^{j\omega})$? If not, what transformation could we apply to the signal to make this true?

[1]Adapted from Problem 2.55 of *Discrete-Time Signal Processing*, 3rd edition, by Oppenheim and Schafer.

Problem 4: DTFT Properties for Salient Time-Domain Features Let $x[n]$ be the signal whose Fourier Transform $X(e^{j\omega})$ is shown below over $[-\pi, \pi]$ and 2π-periodically replicates outside this region.



Regarding the discontinuity, assume $X(e^{j\omega})|_{\omega=0} = 0$. Without computing $x[n]$, determine the following:

(a) $x[0]$.

(b) $\sum_{n=-\infty}^{\infty} x[n]$.

(c) $\sum_{n=-\infty}^{\infty} |x[n]|^2$.

(d) An expression for $x[-n]$ in terms of $x[n]$.

(e) An expression for Re $\{x[n]\}$ in terms of $x[n]$.

(f) An expression for Im $\{x[n]\}$ in terms of $x[n]$.

Problem 5: CTFT for System Analysis [2] A signal $x(t)$ is the input to an LTI system H with impulse response $h(t) = \frac{\sin(500\pi t)}{\pi t}$. Throughout this problem, you may find the Fourier Transform pair

$$s(t) = \frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} S(j\omega) = \begin{cases} 1 & \omega \in [-W, W] \\ 0 & \text{otherwise} \end{cases}$$

to be useful.

(a) Suppose we wanted to calculate the system's frequency response $H(j\omega)$. Without doing any math, and just inspecting $h(t)$, which of the following do we know about $H(j\omega)$? Circle all that apply.

It is purely real	It is purely imaginary
It is even-symmetric	It is odd-symmetric

(b) Provide a well-labeled plot of the system's frequency response $H(j\omega)$.

(c) For each signal $y(t)$ below, determine whether or not there exists an input $x(t)$ to the system H that could produce $y(t)$ as an output. You do **not** need to determine what inputs could produce these outputs, where possible, just whether or not at least one such input exists.

(i) $y(t) = \delta(t)$.

(ii) $y(t) = \cos(100\pi t)$.

(iii) $y(t) = \cos(375\pi t)$.

(iv) $y(t) = \sin(600\pi t)$. Would your answer change if the signal was a cosine at the same frequency instead?

(v) $y(t) = \cos(50\pi t)\cos(75\pi t)$.

(vi) $y(t) = \cos(300\pi t)\cos(400\pi t)$.

(vii) $y(t) = 12e^{j300\pi t}$.

(viii) $y(t) = e^{-7t}u(t)$.

[2] Extension of Problem 1e of EE 120 Fall 2007 Midterm 1.

(d) Now, consider the signal

$$x(t) = x_1(t) + x_2(t)$$

where

$$x_1(t) = \cos(2000\pi t) \text{ and } x_2(t) = \frac{\sin(1500\pi t)}{\pi t}$$

which is input to the LTI system H.

(i) Provide a well-labelled plot of $X(j\omega)$, the Fourier Transform of $x(t)$.

(ii) Provide a well-labelled plot of $Y(j\omega)$, the Fourier Transform of $y(t)$, the output of the LTI system for the signal $x(t)$.

(iii) Write an expression for $y(t)$.

(iv) Would your answer to the previous part change at all if we had used $x_2(t)$, rather than $x(t)$, as our input? Why or why not?

(e) Calculate the energy

$$E_y = \int_{-\infty}^{\infty} |y(t)|^2 dt$$

of the output signal $y(t)$ you found in the previous part.

Problem 6: CTFT for Heat Flow [3] In 1822, Joseph Fourier published *Théorie analytique de la chaleur (The Analytical Theory of Heat)*, in which he developed and solved the heat equation, laying the groundwork for the same Fourier Transform we use today.

Suppose we have a very long rod (one that is essentially infinite in length, so that boundary conditions can be ignored). Let the temperature at position $x \in \mathbb{R}$ along the rod at time $t \geq 0$ be given by $u(x,t)$. The heat equation says that the temperature changes according to

$$\frac{\partial}{\partial t} u(x,t) = \alpha \frac{\partial^2}{\partial x^2} u(x,t) \tag{1}$$

where $\alpha > 0$ is a constant that depends on the thermal diffusivity of the rod's material. As an initial condition, the temperature along any point of the rod at time zero is given as $u(x,0) = g(x)$.

In this problem, we will use the Fourier Transform to solve the heat equation. However, we will be taking the Fourier Transform with respect to the spatial variable x rather than the time variable t as we typically do. To this end, define

$$U(j\omega, t) = \int_{-\infty}^{\infty} u(x,t) e^{-j\omega x} dx \tag{2}$$

so that

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(j\omega, t) e^{j\omega x} d\omega \tag{3}$$

Note that instead of working back and forth between the time and temporal frequency domains as we usually do, we're now working between the space and spatial frequency domains. Fortunately, all the same principles and properties you're familiar with carry over.

[3] Adapted from Problem 5 of EE 120 Spring 2016 Midterm 1.

(a) Show that $\frac{\partial}{\partial t}U(j\omega,t) = -\alpha\omega^2 U(j\omega,t)$.

Hint: Think of ω as some fixed, real constant for this part. Start by plugging (2) into the left hand side of the expression you're asked to show, and then use the heat equation combined with the time (here, space) differentiation property.

(b) Now, show that

$$\frac{\partial}{\partial t}\left[e^{\alpha\omega^2 t}U(j\omega,t)\right] = e^{\alpha\omega^2 t}\left(\frac{\partial}{\partial t}U(j\omega,t) + \alpha\omega^2 U(j\omega,t)\right) \tag{4}$$

(c) Use the results of the previous two parts to show

$$\frac{\partial}{\partial t}\left[e^{\alpha\omega^2 t}U(j\omega,t)\right] = 0 \tag{5}$$

(d) Integrate both sides of the result of the previous part with respect to t, and then apply the initial condition given to show that

$$U(j\omega,t) = G(j\omega)e^{-\alpha\omega^2 t} \tag{6}$$

where $G(j\omega)$ is the spatial Fourier Transform of $g(x) = u(x,0)$.

(e) Express $u(x,t)$ as a convolution integral in terms of the signal g. You may find the Fourier Transform pair

$$\frac{1}{2\sqrt{\pi\kappa}}e^{-x^2/4\kappa} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\kappa\omega^2}$$

to be useful. Here, $\kappa > 0$ is some real constant, and x is the variable over which the transform is taken.

(f) Given the initial condition $u(x,0) = \delta(x)$, solve for $u(x,t)$ using your result in the previous part.[4]

(g) Determine an expression for, and provide well-labelled plots of, $u(x, t = \frac{1}{2\alpha})$ and $u(x, t = \frac{3}{2\alpha})$ on the same graph. What happens to the spatial distribution of the heat in the rod as time goes on?

[4]If you like, you can think of the integral of the rod's temperature over all space, $\int_{-\infty}^{\infty} u(x,t)dx$, as representing the total amount of heat in it at time t. This isn't quite correct due to units not working out, but it's a passable analogy for this problem. Using this metaphor, the initial condition $g(x) = \delta(x)$ means that we have one unit of heat placed in the middle of the rod at time $t = 0$, and we want to see how it spreads out.